UNIT III ARCHES

Arches as structural forms – Examples of arch structures – Types of arches – Analysis of three hinged, two hinged and fixed arches, parabolic and circular arches – Settlement and temperature effects.

Instructional Objectives:

After reading this chapter the student will be able to
1. Differentiate between rigid and deformable structures.
2. Define funicular structure.
3. State the type stress in a cable.
4. Analyse cables subjected to uniformly distributed load.
5. Analyse cables subjected to concentrated loads.

3.1.1 Introduction

Cables and arches are closely related to each other and hence they are grouped in this course in the same module. For long span structures (for e.g. in case bridges) engineers commonly use cable or arch construction due to their efficiency. In the first lesson of this module, cables subjected to uniform and concentrated loads are discussed. In the second lesson, arches in general and three hinged arches in particular along with illustrative examples are explained.
In the last two lessons of this module, two hinged arch and hingeless arches are considered. Structure may be classified into rigid and deformable structures depending on change in geometry of the structure while supporting the load. Rigid structures support externally applied loads without appreciable change in their shape (geometry). Beams trusses and frames are examples of rigid structures. Unlike rigid structures, deformable structures undergo changes in their shape according to externally applied loads. However, it should be noted that deformations are still small. Cables and fabric structures are deformable structures. Cables are mainly used to support suspension roofs, bridges and cable car system. They are also used in electrical transmission lines and for structures supporting radio antennas. In the following sections, cables subjected to concentrated load and cables subjected to uniform loads are considered.

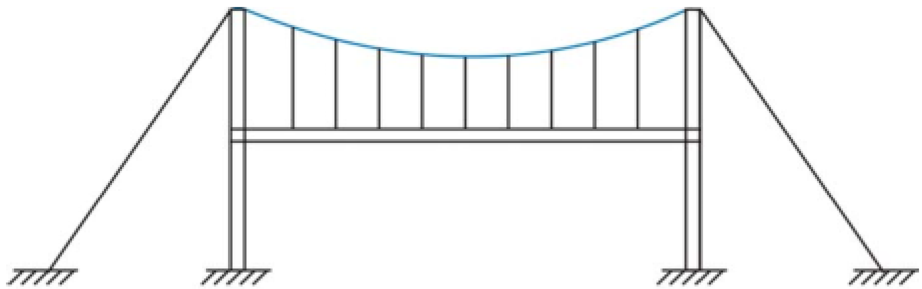


Fig. 31.1 Deformable structure.

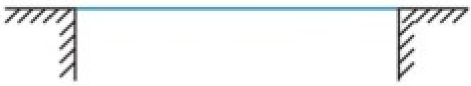


Fig 31.2a Unloaded cable (when dead load is neglected)

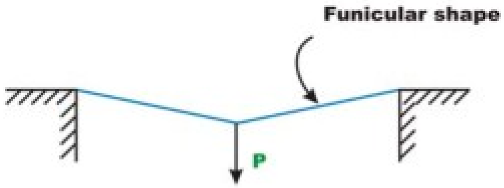


Figure 31.2b Cable in tension.

The shape assumed by a rope or a chain (with no stiffness) under the action of external loads when hung from two supports is known as a funicular shape. Cable is a funicular structure. It is easy to visualize that a cable hung from two supports subjected to external load must be in tension (vide Fig. 31.2a and 31.2b). Now let us modify our definition of cable. A cable may be defined as the structure in pure tension having the funicular shape of the load.

31.2 Cable subjected to Concentrated Loads

As stated earlier, the cables are considered to be perfectly flexible (no flexural stiffness) and inextensible. As they are flexible they do not resist shear force and bending moment. It is subjected to axial tension only and it is always acting tangential to the cable at any point along the length. If the weight of the cable is negligible as compared with the externally applied loads then its self weight is neglected in the analysis. In the present analysis self weight is not considered.

Consider a cable $ACDEB$ as loaded in Fig. 31.2. Let us assume that the cable lengths L_1, L_2, L_3, L_4 and sag at C, D, E (h_c, h_d, h_e) are known. The four reaction components at A and B, cable tensions in each of the four segments and three sag values: a total of eleven unknown quantities are to be determined. From the geometry, one could write two force equilibrium equations ($\sum F_x = 0, \sum F_y = 0$) at each of the point A, B, C, D and E i.e. a total of ten equations and the required one more equation may be written from the geometry of the cable. For example, if one of the sag is given then the problem can be solved easily. Otherwise if the total length of the cable S is given then the required equation may be written as

$$S = \sqrt{L_1^2 + h_c^2} + \sqrt{L_2^2 + (h_d - h_c)^2} + \sqrt{L_2^2 + (h_d - h_e)^2} + \sqrt{L_2^2 + (h + h_e)^2} \qquad (31.1)$$

31.3 Cable subjected to uniform load.

Cables are used to support the dead weight and live loads of the bridge decks having long spans. The bridge decks are suspended from the cable using the hangers. The stiffened deck prevents the supporting cable from changing its shape by distributing the live load moving over it, for a longer length of cable. In such cases cable is assumed to be uniformly loaded.

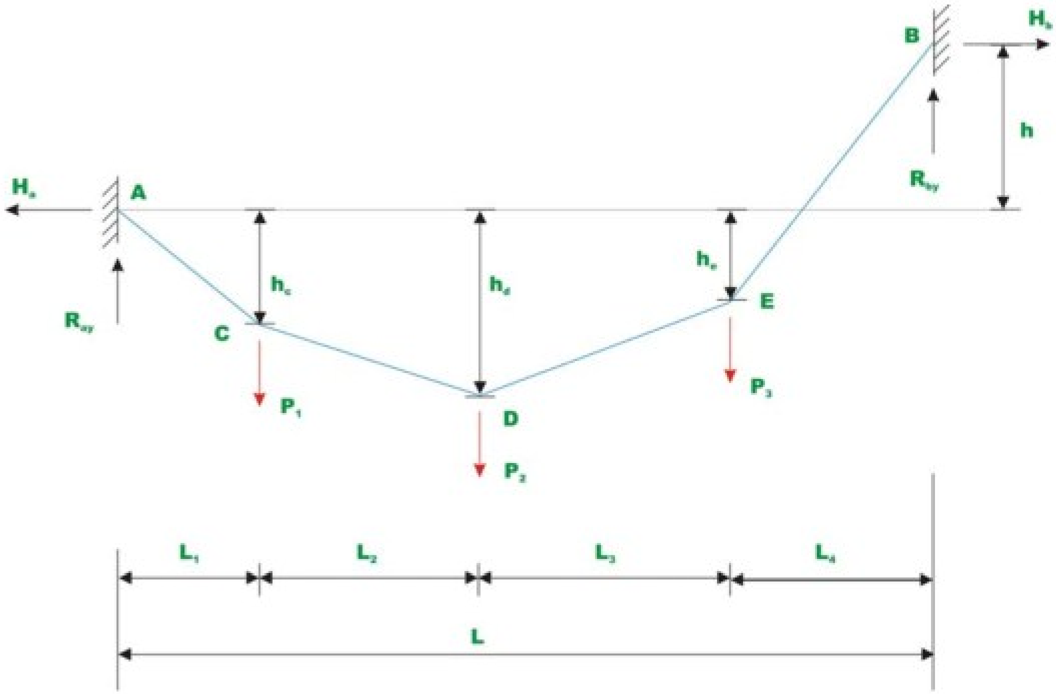
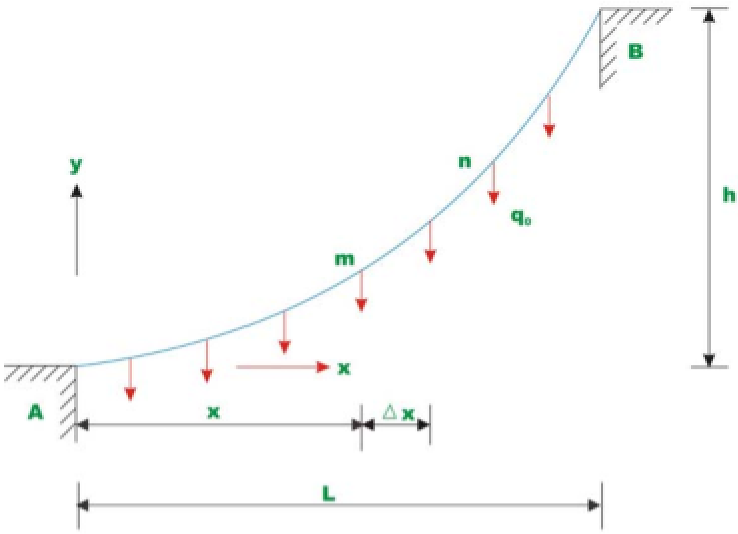


Fig. 31.3a Cable subjected to concentrated load.



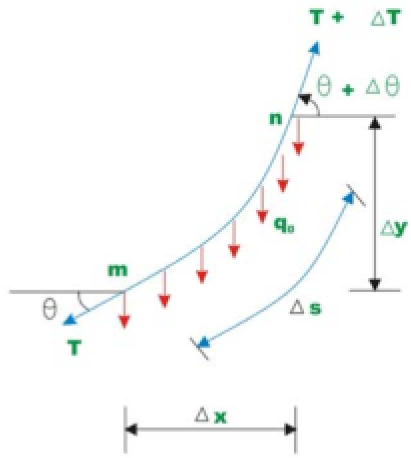


Fig. 31.3b Cable subjected to uniformly distributed load. **Fig. 31.3c Free-body diagram**

Consider a cable which is uniformly loaded as shown in Fig 31.3a. Let the slope of the cable be zero at A. Let us determine the shape of the cable subjected to uniformly distributed load q_0. Consider a free body diagram of the cable as shown in Fig 31.3b. As the cable is uniformly loaded, the tension in the cable changes continuously along the cable length. Let the tension in the cable at m end of the free body diagram be T and tension at the n end of the cable be $T + \Delta T$. The slopes of the cable at m and n are denoted by θ and $\theta + \Delta\theta$ respectively. Applying equations of equilibrium, we get

$$\sum Fy = 0 \qquad -T\sin\theta + (T + \Delta T)\sin(\theta + \Delta\theta) - q_0(\Delta x) = 0 \qquad (31.2a)$$

$$\sum Fx = 0 \qquad -T\cos\theta + (T + \Delta T)\cos(\theta + \Delta\theta) = 0 \qquad (31.2b)$$

$$\sum Mn = 0 \qquad -(T\cos\theta)\Delta y + (T\sin\theta)\Delta x + (q_0\Delta x)\frac{\Delta x}{2} = 0 \qquad (31.2c)$$

Dividing equations 31.2a, b, c by Δx and noting that in the limit as $\Delta x \to 0, \Delta y \to 0 \ \ \Delta\theta \to 0$ and $\Delta T \to 0$.

$$\lim_{\Delta x \to 0} \quad \frac{\Delta T}{\Delta x}\sin(\theta + \Delta\theta) = q_0$$

$$\frac{d}{dx}(T\sin\theta) = q_0 \qquad (31.3a)$$

$$\frac{d}{dx}(T\cos\theta) = 0 \qquad (31.3b)$$

$$\lim_{\Delta x \to 0} \quad -T\cos\theta\frac{\Delta y}{\Delta x} + T\sin\theta + q_0\frac{x_0}{2} = 0$$

$$\frac{dy}{dx} = \tan\theta \qquad (31.3c)$$

Integrating equation (31.3b) we get

$$T\cos\theta = \text{constant}$$

At support (i.e., at $x = 0$), $\quad T\cos\theta = H \qquad$ (31.4a)

i.e. horizontal component of the force along the length of the cable is constant.

Integrating equation 31.3a,

$$T\sin\theta = q_0 x + C_1$$

At $x = 0,\quad T\sin\theta = 0,\quad C_1 = 0\quad$ as $\theta = 0$ at that point.

Hence, $\quad T\sin\theta = q_0 x \qquad$ (31.4b)

From equations 31.4a and 31.4b, one could write

$$\tan\theta = \frac{q_0 x}{H} \qquad (31.4c)$$

From the figure, $\quad \tan\theta = \frac{dy}{dx} = \frac{q_0 x}{H}$

$$\therefore y = \frac{q_0 x^2}{2H} + C$$

At $x = 0, y = 0 \Rightarrow C = 0$ and $y = \dfrac{q_0 x^2}{2H}$ (31.5)

Equation 31.5 represents a parabola. Now the tension in the cable may be evaluated from equations 31.4a and 31.4b. i.e,

$$T = \sqrt{q_0^{\,2} x^2 + H^2}$$

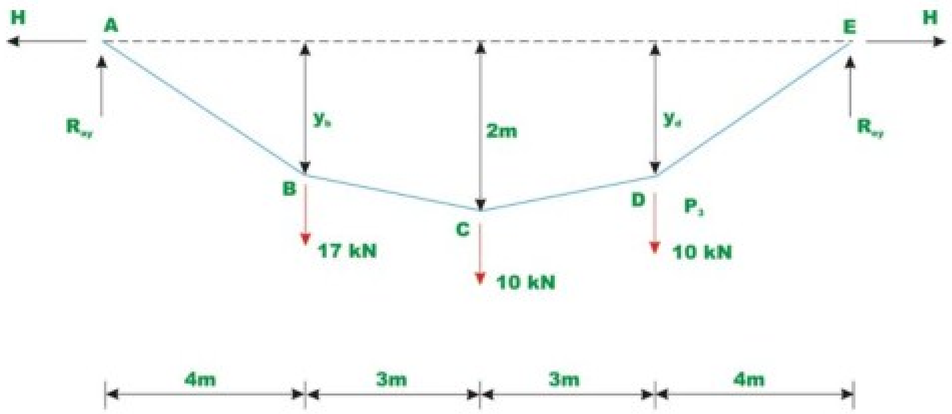
$T = T_{max}$, when $x = L$.

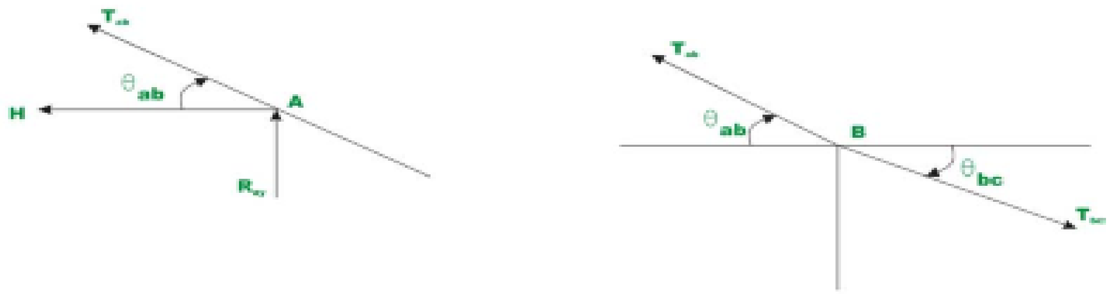
$$T_{max} = \sqrt{q_0^{\,2} L^2 + H^2} = q_0 L \sqrt{1 + \left(\frac{H}{q_0 L}\right)^2} \qquad (31.6)$$

Due to uniformly distributed load, the cable takes a parabolic shape. However due to its own dead weight it takes a shape of a catenary. However dead weight of the cable is neglected in the present analysis.

Example 31.1

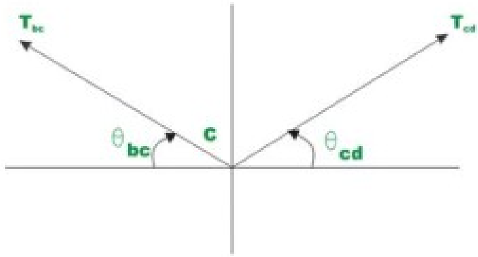
Determine reaction components at A and B, tension in the cable and the sag y_B, and y_E of the cable shown in Fig. 31.4a. Neglect the self weight of the cable in the analysis.



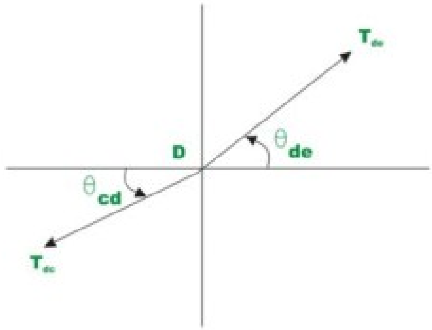


Joint A

Joint B



Joint C



Joint D

Since there are no horizontal loads, horizontal reactions at A and B should be the same. Taking moment about *E*, yields

$$R_{ay} \times 14 - 17 \times 20 - 10 \times 7 - 10 \times 4 = 0$$

$$R_{ay} = \frac{280}{14} = 20 \quad \text{kN}; \qquad R_{ey} = 37 - 20 = 17\text{kN}.$$

Now horizontal reaction H may be evaluated taking moment about point C of all forces left of C.

$$R_{ay} \times 7 - H \times 2 - 17 \times 3 = 0$$

$$H = 44.5 \text{ kN}$$

Taking moment about B of all the forces left of B and setting $M_B = 0$, we get

$$R_{ay} \times 4 - H \times y_B = 0; \qquad y_B = \frac{80}{44.50} = 1.798\, m$$

Similarly, $y_D = \dfrac{68}{44.50} = 1.528\, m$

To determine the tension in the cable in the segment AB, consider the equilibrium of joint A (vide Fig.31.4b).

$$\sum F_x = 0 \Rightarrow T_{ab} \cos\theta_{ab} = H$$

$$T_{ab} = \frac{44.5}{\left(\frac{3}{\sqrt{3^2 + 0.298^2}} \right)} = 48.789 \text{ kN}$$

The tension T_{ab} may also be obtained as

$$T_{ab} = \sqrt{R_{ay}^{\ 2} + H^2} = \sqrt{20^2 + 44.5^2} = 48.789 \ \ \text{kN}$$

Now considering equilibrium of joint $B, C,$ and D one could calculate tension in different segments of the cable.

Segment bc

Applying equations of equilibrium,

$$\sum F_x = 0 \Rightarrow T_{ab} \cos\theta_{ab} = T_{bc} \cos\theta_{bc}$$

$$T_{bc} = \frac{44.5}{\left(\frac{3}{\sqrt{3^2 + 0.298^2}} \right)} \cong 44.6 \ \ \text{kN}$$

See Fig.31.4c

Segment cd

$$T_{cd} = \frac{T_{bc} \cos\theta_{bc}}{\cos\theta_{cd}} = \frac{44.5}{\left(\frac{3}{\sqrt{3^2 + 0.472^2}} \right)} = 45.05 \ \ \text{kN}$$

See Fig.31.4d.
See Fig.31.4e.

Segment de

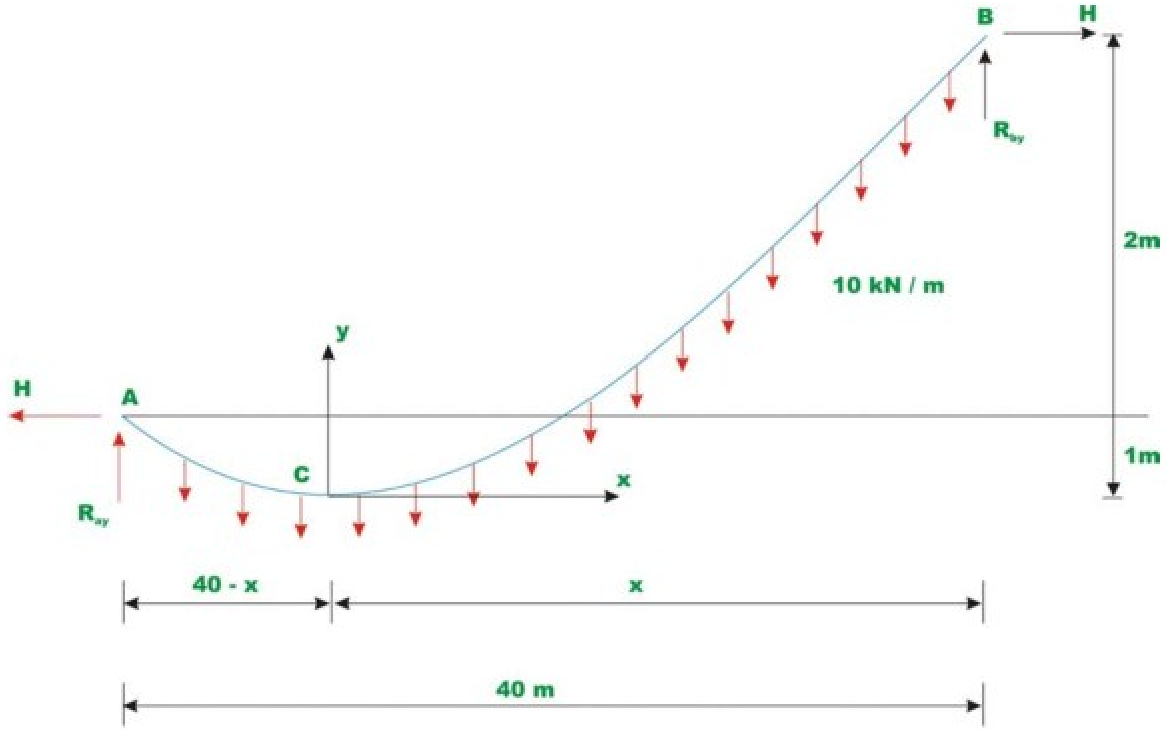
$$T_{de} = \frac{T_{cd} \cos\theta_{cd}}{\cos\theta_{de}} = \frac{44.5}{\frac{4}{\sqrt{4^2 + 1.528^2}}} = 47.636 \ \ \text{kN}$$

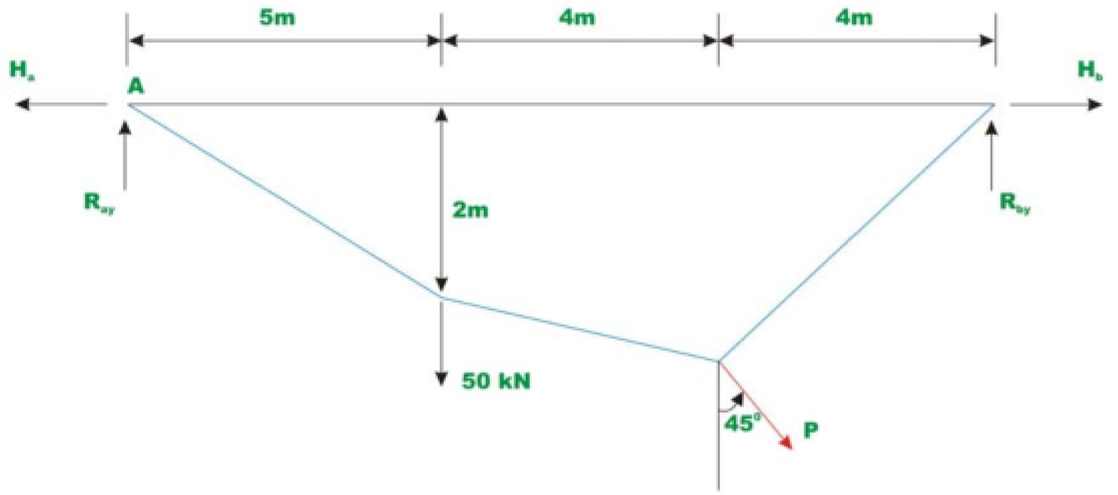
The tension T_{de} may also be obtained as

$$T_{de} = \sqrt{R_{ey}^{\,2} + H^2} = \sqrt{17^2 + 44.5^2} = 47.636 \quad \text{kN}$$

Example 31.2

A cable of uniform cross section is used to span a distance of 40m as shown in Fig 31.5. The cable is subjected to uniformly distributed load of 10 kN/m. run. The left support is below the right support by 2 m and the lowest point on the cable C is located below left support by 1 m. Evaluate the reactions and the maximum and minimum values of tension in the cable.





Assume the lowest point C to be at distance of x m from B. Let us place our origin of the co-ordinate system xy at C. Using equation 31.5, one could write,

$$y_a = 1 = \frac{q_0 (40 - x)^2}{2H} = \frac{10(40 - x)^2}{2H} \tag{1}$$

$$y_b = 3 = \frac{10x^2}{2H} \tag{2}$$

where y_a and y_b be the y co-ordinates of supports A and B respectively. From equations 1 and 2, one could evaluate the value of x. Thus,

$$10(40 - x)^2 = \frac{10x^2}{3} \quad \Rightarrow \quad x = 25.359\, m$$

From equation 2, the horizontal reaction can be determined.

$$H = \frac{10 \times 25.359^2}{6} = 1071.80 \text{ kN}$$

Now taking moment about A of all the forces acting on the cable, yields

$$R_{by} = \frac{10 \times 40 \times 20 + 1071.80 \times 2}{40} = 253.59 \text{ kN}$$

Writing equation of moment equilibrium at point B, yields

$$R_{ay} = \frac{40 \times 20 \times 10 - 1071.80 \times 2}{40} = 146.41 \text{ kN}$$

Tension in the cable at supports A and B are

$$T_A = \sqrt{146.41^2 + 1071.81^2} = 1081.76 \text{ kN}$$

$$T_B = \sqrt{253.59^2 + 1071.81^2} = 1101.40 \text{ kN}$$

The tension in the cable is maximum where the slope is maximum as $T\cos\theta = H$. The maximum cable tension occurs at B and the minimum cable tension occurs at C where $\frac{dy}{dx} = \theta = 0$ and $T_C = H = 1071.81$ kN

Example 31.3

A cable of uniform cross section is used to support the loading shown in Fig 31.6. Determine the reactions at two supports and the unknown sag y_C.

Taking moment of all the forces about support B,

$$R_{ay} = \frac{1}{10}\left[350 + 300 + 100 y_c\right] \quad (1)$$

$$R_{ay} = 65 + 10 y_c$$

Taking moment about B of all the forces left of B and setting $M_B = 0$, we get,

$$R_{ay} \times 3 - H_a \times 2 = 0$$
$$\Rightarrow H_a = 1.5 R_{ay} \quad (2)$$

Taking moment about C of all the forces left of C and setting $M_C = 0$, we get

$$\sum M_C = 0 \qquad R_{ay} \times 7 - H_a \times y_C - 50 \times 4 = 0$$

Substituting the value of H_a in terms of R_{ay} in the above equation,

$$7 R_{ay} - 1.5 R_{ay} y_C - 200 = 0 \quad (3)$$

Using equation (1), the above equation may be written as,

$$y_C^2 + 1.833 y_C - 17 = 0 \quad (4)$$

Solving the above quadratic equation, y_C can be evaluated. Hence,

$$y_C = 3.307 m.$$

Substituting the value of y_C in equation (1),

$$R_{ay} = 98.07 \text{ kN}$$

From equation (2),

$$H_a = 1.5 R_{ay} = 147.05 \text{ kN}$$

Now the vertical reaction at D, R_{dy} is calculated by taking moment of all the forces about A,

$$R_{dy} \times 10 - 100 \times 7 + 100 \times 3.307 - 50 \times 3 = 0$$

$$R_{dy} = 51.93 \text{ kN.}$$

Taking moment of all the forces right of C about C, and noting that $\sum M_C = 0$,

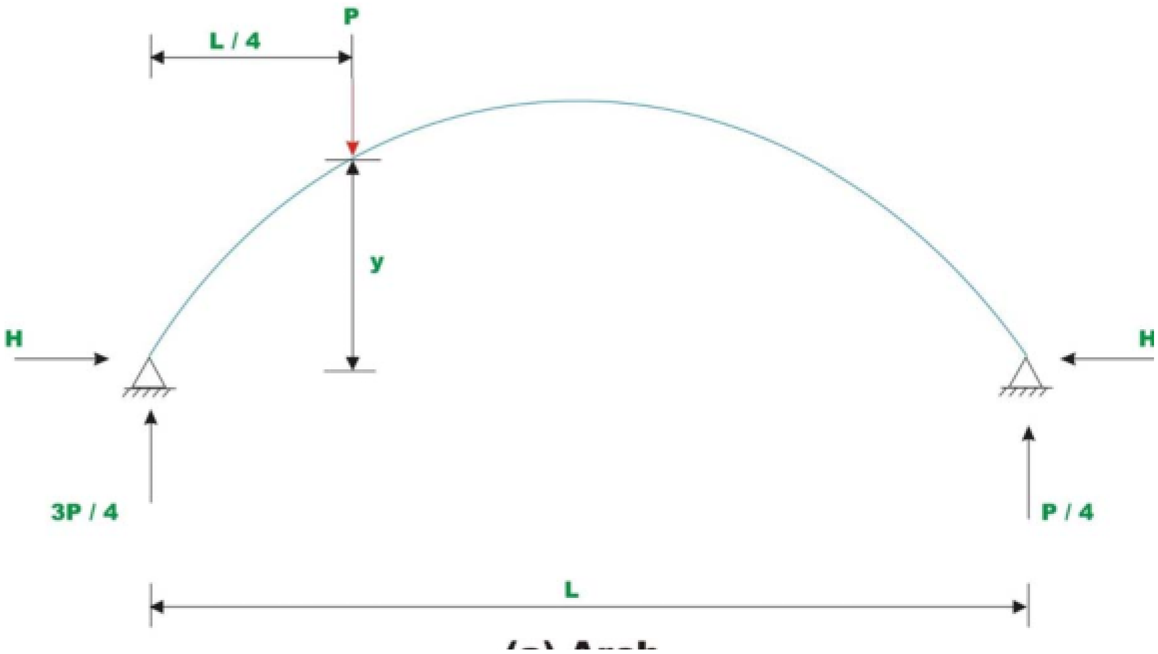
$$R_{dy} \times 3 = H_d \times y_C \quad \Rightarrow H_d = 47.109 \text{ kN.}$$

Instructional Objectives:

After reading this chapter the student will be able to
1. Define an arch.
2. Identify three-hinged, two-hinged and hingeless arches.
3. State advantages of arch construction.
4. Analyse three-hinged arch.
5. Evaluate horizontal thrust in three-hinged arch.

32.1 Introduction

In case of beams supporting uniformly distributed load, the maximum bending moment increases with the square of the span and hence they become uneconomical for long span structures. In such situations arches could be advantageously employed, as they would develop horizontal reactions, which in turn reduce the design bending moment.



Beam and Arch comparison.

For example, in the case of a simply supported beam shown in Fig. 32.1, the bending moment below the load is $\frac{3PL}{16}$. Now consider a two hinged symmetrical arch of the same span and subjected to similar loading as that of simply supported beam. The vertical reaction could be calculated by equations of statics. The horizontal reaction is determined by the method of least work. Now the bending moment below the load is $\frac{3PL}{16} - Hy$. It is clear that the bending moment below the load is reduced in the case of an arch as compared to a simply supported beam. It is observed in the last lesson that, the cable takes the shape of the loading and this shape is termed as funicular shape. If an arch were constructed in an inverted funicular shape then it would be subjected to only compression for those loadings for which its shape is inverted funicular.

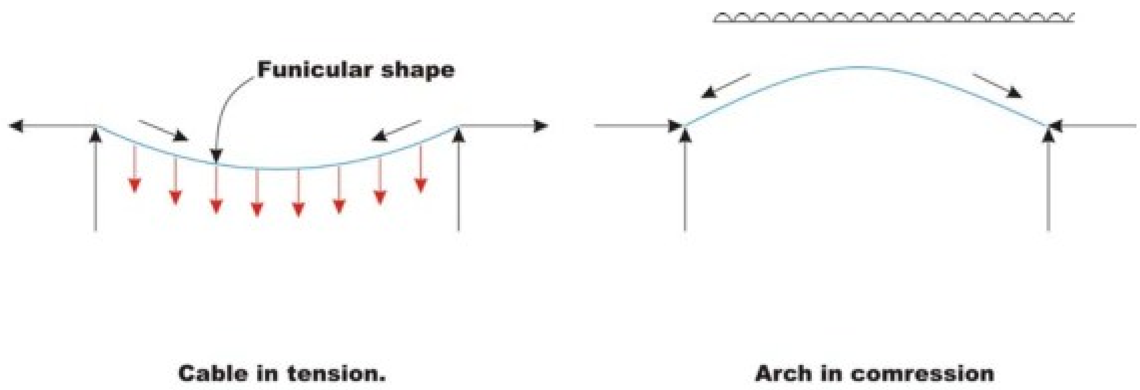
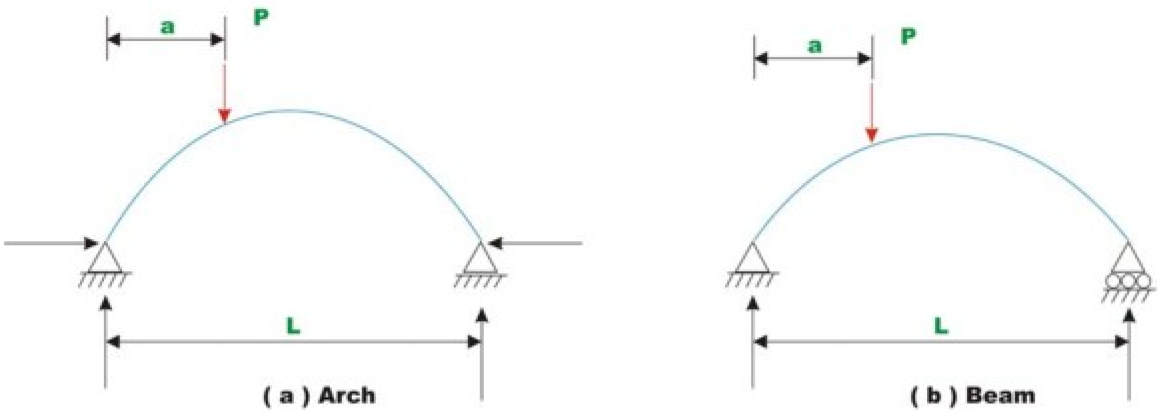


Fig. 32.2 Cable and Arch structure.

Since in practice, the actual shape of the arch differs from the inverted funicular shape or the loading differs from the one for which the arch is an inverted funicular, arches are also subjected to bending moment in addition to compression. As arches are subjected to compression, it must be designed to resist buckling.

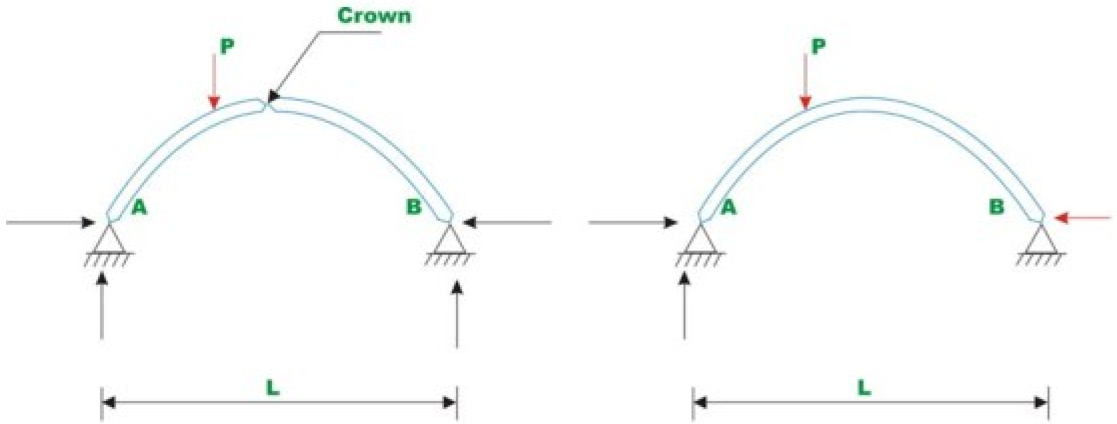
Until the beginning of the 20th century, arches and vaults were commonly used to span between walls, piers or other supports. Now, arches are mainly used in bridge construction and doorways. In earlier days arches were constructed using stones and bricks. In modern times they are being constructed of reinforced concrete and steel.



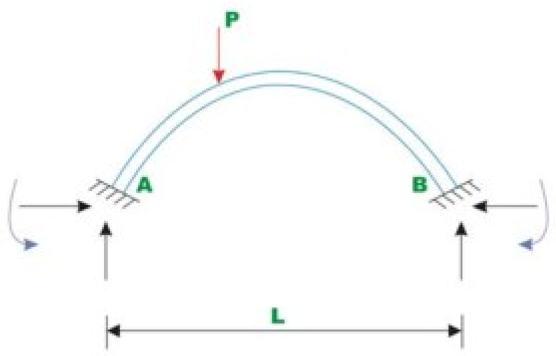
A structure is classified as an arch not based on its shape but the way it supports the lateral load. Arches support load primarily in compression. For example in Fig 32.3b, no horizontal reaction is developed. Consequently bending moment is not reduced. It is important to appreciate the point that the definition of an arch is a structural one, not geometrical.

32.2 Type of arches

There are mainly three types of arches that are commonly used in practice: three hinged arch, two-hinged arch and fixed-fixed arch. Three-hinged arch is statically determinate structure and its reactions / internal forces are evaluated by static equations of equilibrium. Two-hinged arch and fixed-fixed arch are statically indeterminate structures. The indeterminate reactions are determined by the method of least work or by the flexibility matrix method. In this lesson three-hinged arch is discussed.



(a) Three - hinged arch **(b) Two - hinged arch**



(c) Fixed hinged arch

32.3 Analysis of three-hinged arch

In the case of three-hinged arch, we have three hinges: two at the support and one at the crown thus making it statically determinate structure. Consider a three hinged arch subjected to a concentrated force P as shown in Fig 32.5.

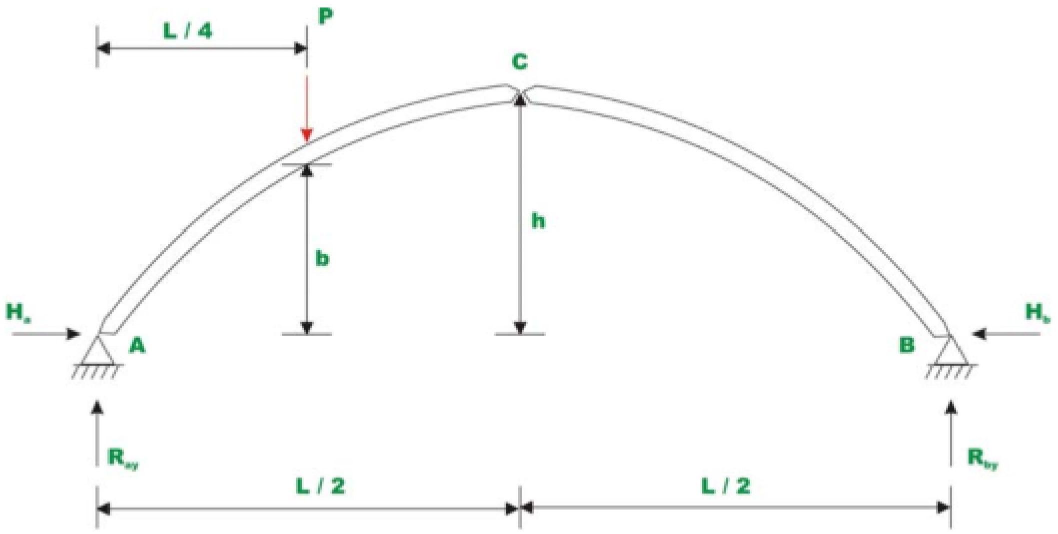


Fig. 32.5 Three hinged arch.

There are four reaction components in the three-hinged arch. One more equation is required in addition to three equations of static equilibrium for evaluating the four reaction components. Taking moment about the hinge of all the forces acting on either side of the hinge can set up the required equation. Taking moment of all the forces about hinge A, yields

$$R_{by} = \frac{PL}{4L} = \frac{P}{4} \tag{32.1}$$

$$\sum Fy = 0 \qquad \Rightarrow \qquad R_{ay} = \frac{3P}{4} \tag{32.2}$$

Taking moment of all forces right of hinge C about hinge C leads to

$$H_b \times h = \frac{R_{by}L}{2}$$

$$\Rightarrow \qquad H_b = \frac{R_{by}L}{2h} = \frac{PL}{8h} \tag{32.3}$$

Applying $\sum Fx = 0$ to the whole structure gives $\qquad H_a = \dfrac{PL}{8h}$

Now moment below the load is given by ,

$$M_D = \frac{R_{ay}L}{4} - H_a b$$

$$M_D = \frac{3PL}{16} - \frac{PLb}{8h} \tag{32.4}$$

If $\dfrac{b}{h} = \dfrac{1}{2}$ then $\quad M_D = \dfrac{3PL}{16} - \dfrac{PL}{16} = 0.125PL \qquad (32.5)$

For a simply supported beam of the same span and loading, moment under the loading is given by,

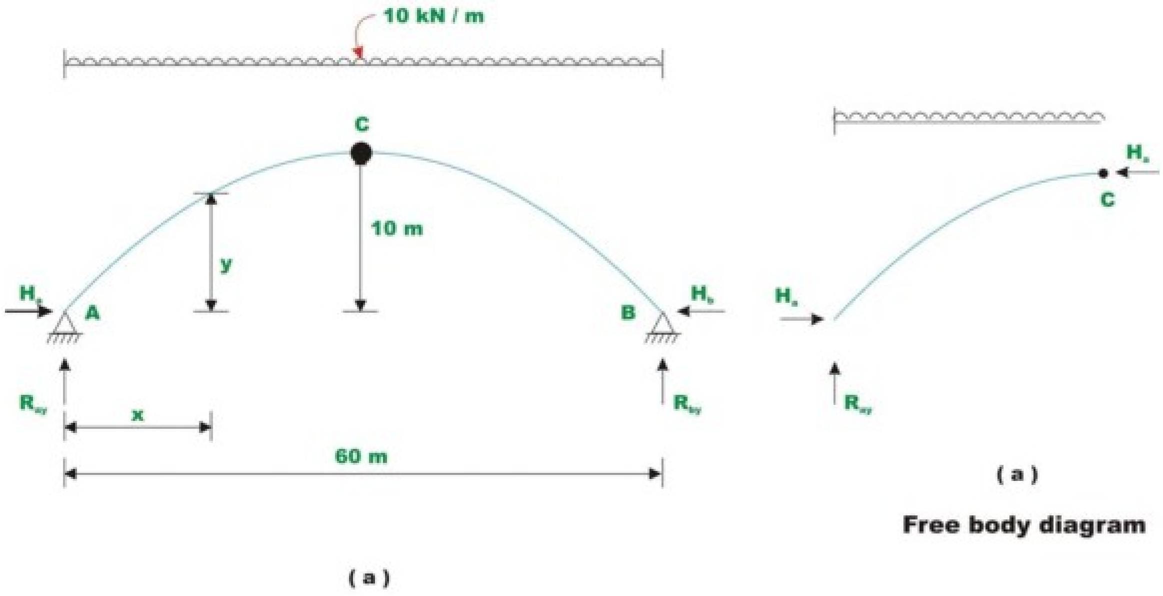
$$M_D = \frac{3PL}{16} = 0.375PL \tag{32.6}$$

For the particular case considered here, the arch construction has reduced the moment by 66.66 %.

Example 32.1

A three-hinged parabolic arch of uniform cross section has a span of 60 m and a rise of 10 m. It is subjected to uniformly distributed load of intensity 10 kN/m as shown in Fig. 32.6 Show that the bending moment is zero at any cross section of the arch.

Solution:



(a)

(a)

Free body diagram

Fig. 32.6 Three hinged arch of Example 32.1

Reactions:

Taking moment of all the forces about hinge A, yields

$$R_{ay} = R_{by} = \frac{10 \times 60}{2} = 300 \ \ \text{kN} \tag{1}$$

Taking moment of forces left of hinge C about C, one gets

$$R_{ay} \times 30 - H_a \times 10 - 10 \times 30 \times \frac{30}{2} = 0$$

$$H_a = \frac{300 \times 30 - 10 \times 30 \times \left(\frac{30}{2}\right)}{10} \tag{2}$$

$$= 450 \ \ \text{kN}$$

From $\sum Fx = 0$ one could write, $H_b = 450 \ \ \text{kN}$.

The shear force at the mid span is zero.

Bending moment

The bending moment at any section x from the left end is,

$$M_x = R_{ay}x - H_a y - 10\frac{x^2}{2} \qquad (3)$$

The equation of the three-hinged parabolic arch is

$$y = \frac{2}{3}x - \frac{10}{30^2}x^2 \qquad (4)$$

$$M_x = 300x - \left(\frac{2}{3}x - \frac{10}{30^2}x^2\right)450 - 5x^2$$

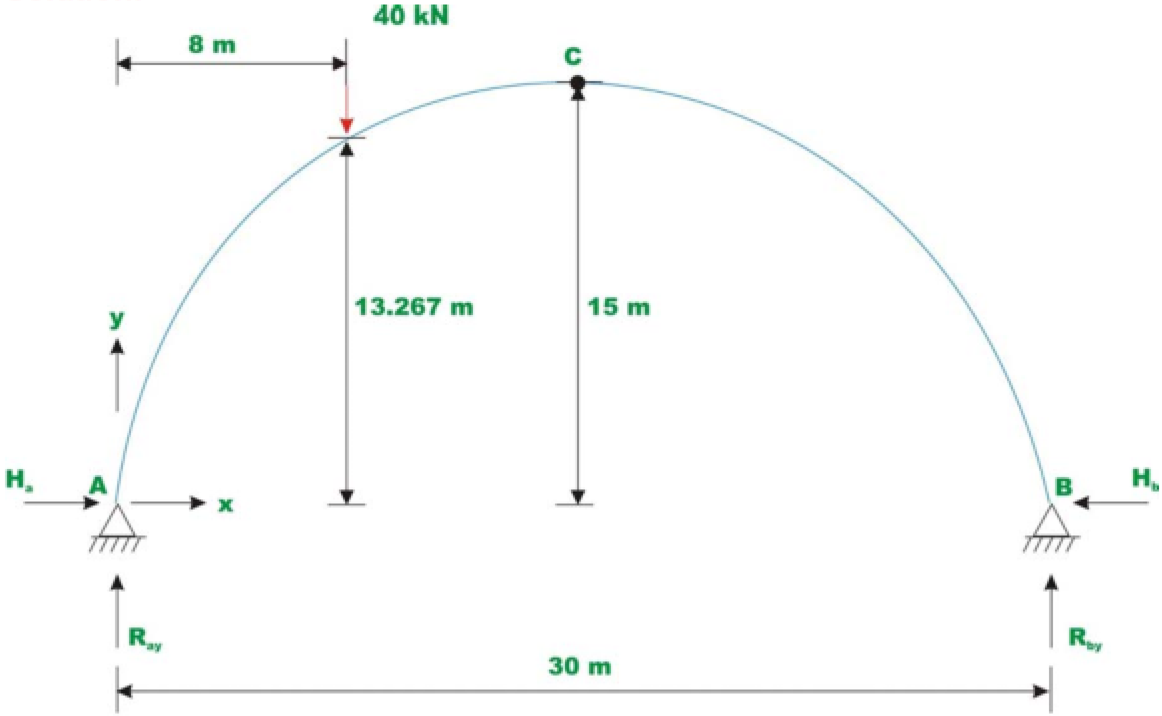
$$= 300x - 300x + 5x^2 - 5x^2 = 0$$

In other words a three hinged parabolic arch subjected to uniformly distributed load is not subjected to bending moment at any cross section. It supports the load in pure compression. Can you explain why the moment is zero at all points in a three-hinged parabolic arch?

Example 32.2

A three-hinged semicircular arch of uniform cross section is loaded as shown in Fig 32.7. Calculate the location and magnitude of maximum bending moment in the arch.

Solution:



Reactions:
Taking moment of all the forces about hinge B leads to,

$$R_{ay} = \frac{40 \times 22}{30} = 29.33 \quad \text{kN} \quad (\uparrow)$$

$$\sum Fy = 0 \qquad \Rightarrow R_{by} = 10.67 \quad \text{kN} \quad (\uparrow) \tag{1}$$

Bending moment
Now making use of the condition that the moment at hinge C of all the forces left of hinge C is zero gives,

$$M_c = R_{ay} \times 15 - H_a \times 15 - 40 \times 7 = 0 \tag{2}$$

$$H_a = \frac{29.33 \times 15 - 40 \times 7}{15} = 10.66 \quad \text{kN} \quad (\rightarrow)$$

Considering the horizontal equilibrium of the arch gives,

$$H_b = 10.66 \quad \text{kN} \quad (\leftarrow)$$

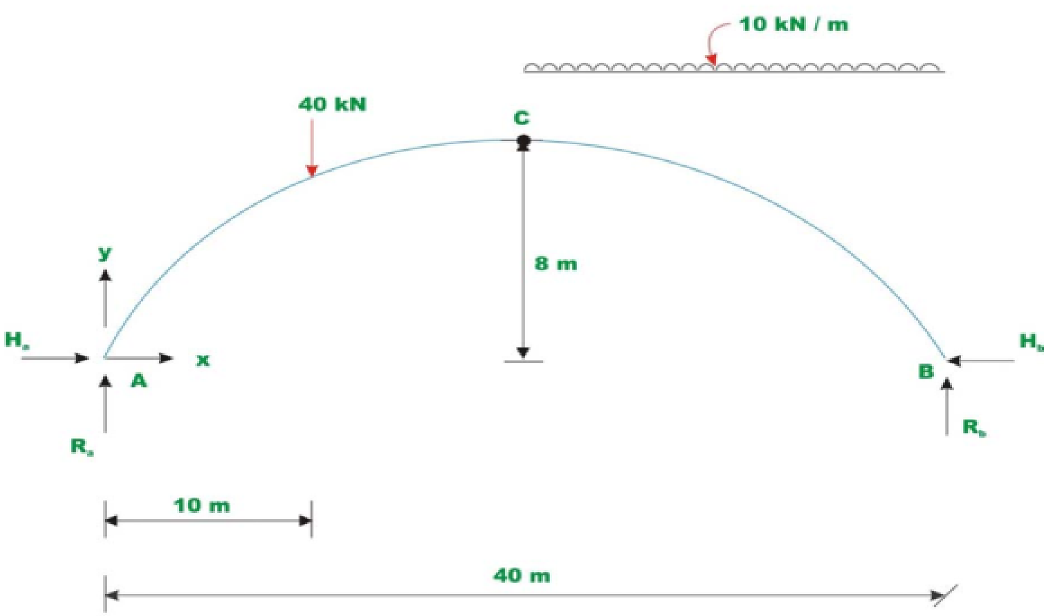
The maximum positive bending moment occurs below D and it can be calculated by taking moment of all forces left of D about D.

$$M_D = R_{ay} \times 8 - H_a \times 13.267 \tag{3}$$

$$= 29.33 \times 8 - 10.66 \times 13.267 = 93.213 \quad \text{kN}$$

Example 32.3
A three-hinged parabolic arch is loaded as shown in Fig 32.8a. Calculate the location and magnitude of maximum bending moment in the arch. Draw bending moment diagram.

Solution:



Reactions:

Taking A as the origin, the equation of the three-hinged parabolic arch is given by,

$$y = \frac{8}{10}x - \frac{8}{400}x^2 \tag{1}$$

Taking moment of all the forces about hinge B leads to,

$$R_{ay} = \frac{40\times 30 + 10\times 20\times\left(20/2\right)}{40} = 80 \quad \text{kN} \quad (\uparrow)$$

$$\sum Fy = 0 \qquad \Rightarrow R_{by} = 160 \quad \text{kN} \quad (\uparrow) \tag{2}$$

Now making use of the condition that, the moment at hinge C of all the forces left of hinge C is zero gives,

$$M_c = R_{ay}\times 20 - H_a\times 8 - 40\times 10 = 0$$

$$H_a = \frac{80\times 20 - 40\times 10}{8} = 150 \quad \text{kN} \quad (\rightarrow) \tag{3}$$

Considering the horizontal equilibrium of the arch gives,

$$H_b = 150 \quad \text{kN} \quad (\leftarrow) \tag{4}$$

Location of maximum bending moment

Consider a section x from end B. Moment at section x in part CB of the arch is given by (please note that B has been taken as the origin for this calculation),

$$M_x = 160x - \left(\frac{8}{10}x - \frac{8}{400}x^2\right)150 - \frac{10}{2}x^2 \tag{5}$$

According to calculus, the necessary condition for extremum (maximum or minimum) is that $\frac{\partial M_x}{\partial x} = 0$.

$$\frac{\partial M_x}{\partial x} = 160 - \left(\frac{8}{10} - \frac{8\times 2}{400}x\right)150 - 10x \tag{6}$$

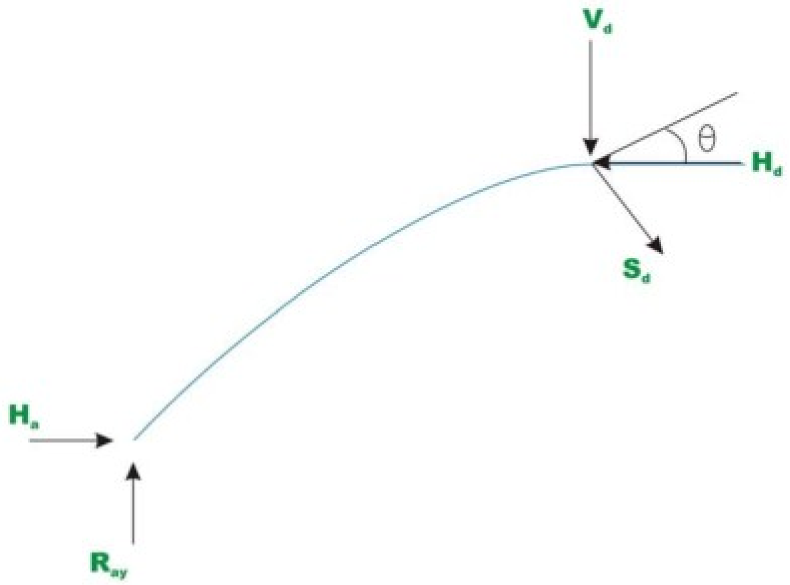
$$= 40 - 4x = 0$$

$$x = 10 \quad \text{m.}$$

Substituting the value of x in equation (5), the maximum bending moment is obtained. Thus,

$$M_{max} = 160(10) - \left(\frac{8}{10}(10) - \frac{8}{400}(10)^2\right)150 - \frac{10}{2}(10)^2$$
$$M_{max} = 200 \text{ kN.m.} \quad (7)$$

Shear force at D just left of 40 kN load



The slope of the arch at D is evaluated by,

$$\tan\theta = \frac{dy}{dx} = \frac{8}{10} - \frac{16}{400}x \quad (8)$$

Substituting $x = 10$ m. in the above equation, $\theta_D = 21.8^0$

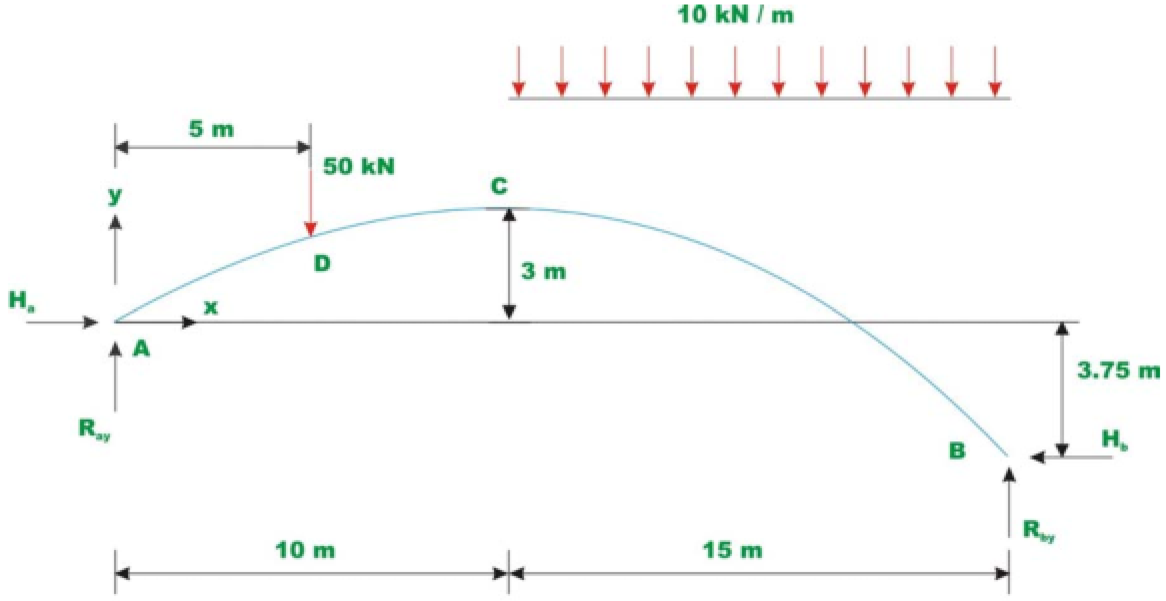
Shear force S_d at left of D is

$$S_d = H_a \sin\theta - R_{ay}\cos\theta \quad (9)$$
$$S_d = 150\sin(21.80) - 80\cos(21.80)$$
$$= -18.57 \text{ kN.}$$

Example 32.4

A three-hinged parabolic arch of constant cross section is subjected to a uniformly distributed load over a part of its span and a concentrated load of 50 kN, as shown in Fig. 32.9. The dimensions of the arch are shown in the figure. Evaluate the horizontal thrust and the maximum bending moment in the arch.

Solution:



Reactions:

Taking A as the origin, the equation of the parabolic arch may be written as,

$$y = -0.03x^2 + 0.6x \tag{1}$$

Taking moment of all the loads about B leads to,

$$\begin{aligned} R_{ay} &= \frac{1}{25}\left[50\times 20 + 10\times 15\times \frac{15}{2} - H_a \times 3.75\right] \\ &= \frac{1}{25}\left[2125 - 3.75H_a\right] \end{aligned} \tag{2}$$

Taking moment of all the forces right of hinge C about the hinge C and setting $M_c = 0$ leads to,

$$R_{by}\times 15 - 6.75H_b - 10\times 15\times \frac{15}{2} = 0$$

$$R_{by} = \frac{1}{15}\left[1125 + 6.75H_b\right] \tag{3}$$

Since there are no horizontal loads acting on the arch,

$$H_a = H_b = H \quad (say)$$

Applying $\sum Fy = 0$ for the whole arch,

$$R_{ay} + R_{by} = 10 \times 15 + 50 = 200$$

$$\frac{1}{25}[2125 - 3.75H] + \frac{1}{15}[1125 + 6.75H] = 200$$

$$85 - 0.15H + 75 + 0.45H = 200$$

$$H = \frac{40}{0.3} = 133.33 \text{ kN} \qquad (4)$$

From equation (2),

$$R_{ay} = 65.0 \text{ kN}$$

$$R_{by} = 135.0 \text{ kN} \qquad (5)$$

Bending moment

From inspection, the maximum negative bending moment occurs in the region AD and the maximum positive bending moment occurs in the region CB.

Span AD

Bending moment at any cross section in the span AD is

$$M = R_{ay}x - H_a(-0.03x^2 + 0.6x) \qquad 0 \le x \le 5 \qquad (6)$$

For, the maximum negative bending moment in this region,

$$\frac{\partial M}{\partial x} = 0 \Rightarrow \qquad R_{ay} - H_a(-0.06x + 0.6) = 0$$

$$x = 1.8748 \text{ m}$$

$$M = -14.06 \text{ kN.m.}$$

For the maximum positive bending moment in this region occurs at D,

$$M_D = R_{ay}5 - H_a(-0.03 \times 25 + 0.6 \times 5)$$

$$= +25.0 \text{ kN.m}$$

Span CB

Bending moment at any cross section, in this span is calculated by,

$$M = R_{ay}x - H_a(-0.03x^2 + 0.6x) - 50(x-5) - 10(x-10)\frac{(x-10)}{2}$$

For locating the position of maximum bending moment,

$$\frac{\partial M}{\partial x} = 0 = R_{ay} - H_a(-0.06x + 0.6) - 50 - \frac{10}{2} \times 2(x - 10) = 0$$

$$x = 17.5 \quad \text{m}$$

$$M = 65 \times 17.5 - 133.33(-0.03(17.5)^2 + 0.6(17.5)) - 50(12.5) - \frac{10}{2}(7.5)^2$$

$$M = 56.25 \quad \text{kN.m}$$

Hence, the maximum positive bending moment occurs in span CB.

Instructional Objectives:

After reading this chapter the student will be able to

1. Compute horizontal reaction in two-hinged arch by the method of least work.
2. Write strain energy stored in two-hinged arch during deformation.
3. Analyse two-hinged arch for external loading.
4. Compute reactions developed in two hinged arch due to temperature loading.

33.1 Introduction

Mainly three types of arches are used in practice: three-hinged, two-hinged and hingeless arches. In the early part of the nineteenth century, three-hinged arches were commonly used for the long span structures as the analysis of such arches could be done with confidence. However, with the development in structural analysis, for long span structures starting from late nineteenth century engineers adopted two-hinged and hingeless arches. Two-hinged arch is the statically indeterminate structure to degree one. Usually, the horizontal reaction is treated as the redundant and is evaluated by the method of least work. In this lesson, the analysis of two-hinged arches is discussed and few problems are solved to illustrate the procedure for calculating the internal forces.

33.2 Analysis of two-hinged arch

A typical two-hinged arch is shown in Fig. 33.1a. In the case of two-hinged arch, we have four unknown reactions, but there are only three equations of equilibrium available. Hence, the degree of statical indeterminacy is one for two-hinged arch.

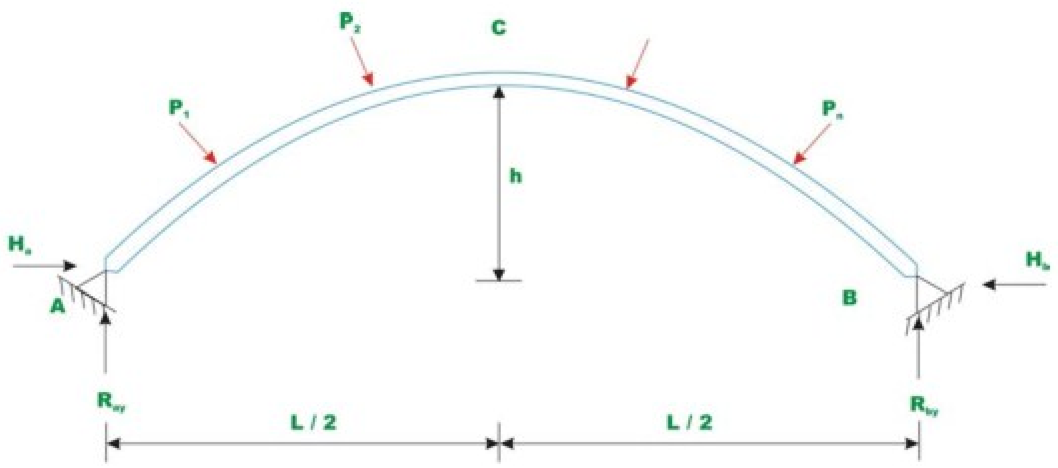
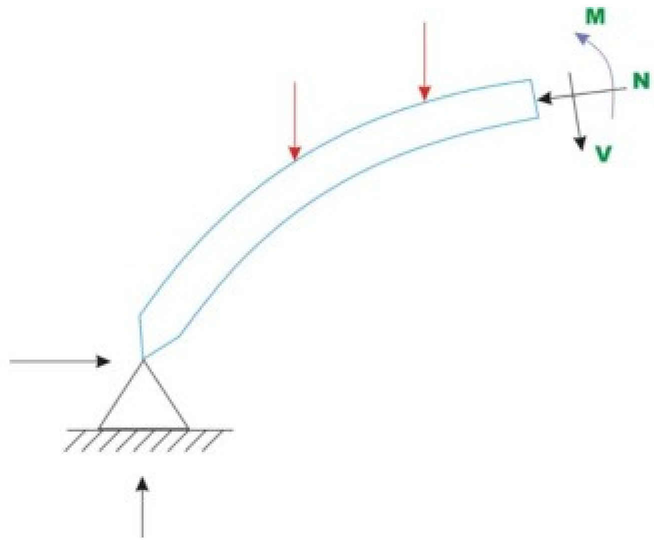


Fig. 33.1a Two - hinged arch.



The fourth equation is written considering deformation of the arch. The unknown redundant reaction H_b is calculated by noting that the horizontal displacement of hinge B is zero. In general the horizontal reaction in the two hinged arch is evaluated by straightforward application of the theorem of least work (see module 1, lesson 4), which states that the partial derivative of the strain energy of a statically indeterminate structure with respect to statically indeterminate action should vanish. Hence to obtain, horizontal reaction, one must develop an expression for strain energy. Typically, any section of the arch (vide Fig 33.1b) is subjected to shear force V, bending moment M and the axial compression N. The strain energy due to bending U_b is calculated from the following expression.

$$U_b = \int_0^s \frac{M^2}{2EI} ds$$

The above expression is similar to the one used in the case of straight beams. However, in this case, the integration needs to be evaluated along the curved arch length. In the above equation, s is the length of the centerline of the arch, I is the moment of inertia of the arch cross section, E is the Young's modulus of the arch material. The strain energy due to shear is small as compared to the strain energy due to bending and is usually neglected in the analysis. In the case of flat arches, the strain energy due to axial compression can be appreciable and is given by,

$$U_a = \int_0^s \frac{N^2}{2AE} ds \qquad (33.2)$$

The total strain energy of the arch is given by,

$$U = \int_0^s \frac{M^2}{2EI} ds + \int_0^s \frac{N^2}{2AE} ds \qquad (33.3)$$

Now, according to the principle of least work

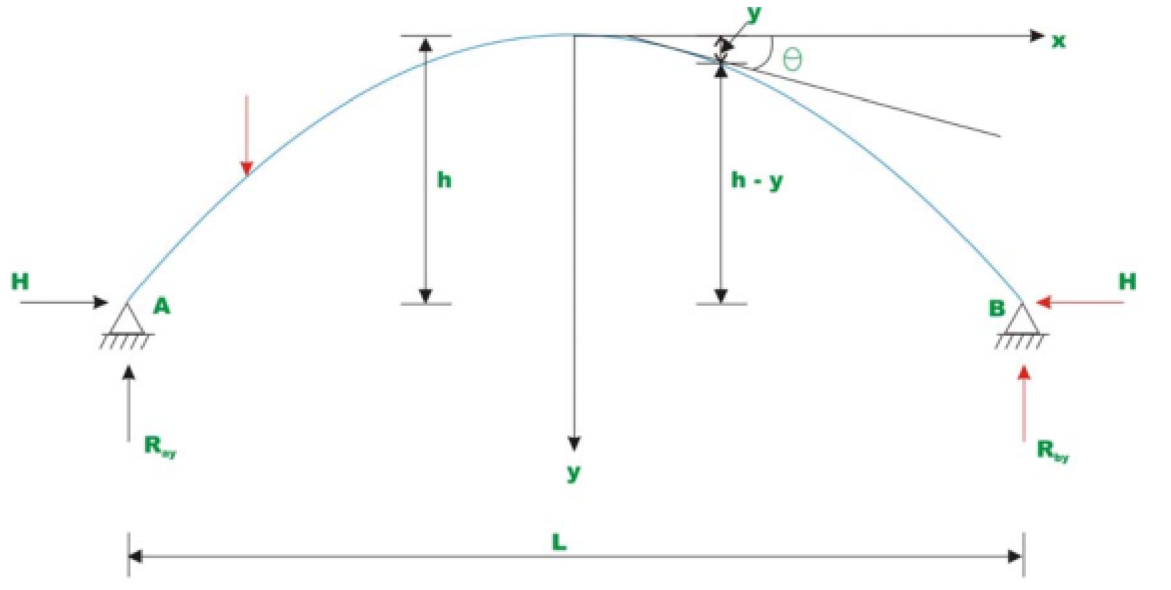
$\frac{\partial U}{\partial H} = 0$, where H is chosen as the redundant reaction.

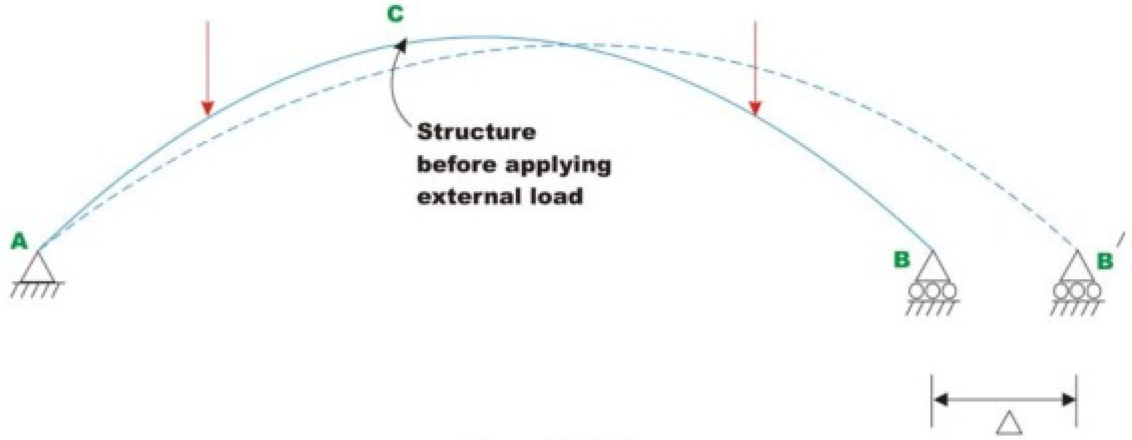
$$\frac{\partial U}{\partial H} = \int_0^s \frac{M}{EI} \frac{\partial M}{\partial H} ds + \int_0^s \frac{N}{AE} \frac{\partial N}{\partial H} ds = 0 \qquad (33.4)$$

Solving equation 33.4, the horizontal reaction H is evaluated.

33.2.1 Symmetrical two hinged arch

Consider a symmetrical two-hinged arch as shown in Fig 33.2a. Let C at crown be the origin of co-ordinate axes. Now, replace hinge at B with a roller support. Then we get a simply supported curved beam as shown in Fig 33.2b. Since the curved beam is free to move horizontally, it will do so as shown by dotted lines in Fig 33.2b. Let M_0 and N_0 be the bending moment and axial force at any cross section of the simply supported curved beam. Since, in the original arch structure, there is no horizontal displacement, now apply a horizontal force H as shown in Fig. 33.2c. The horizontal force H should be of such magnitude, that the displacement at B must vanish.


y
x
θ
h
h - y
H
A
B
H
R_ay
R_by
y
L


C
Structure
before applying
external load
A
B
B'
Δ

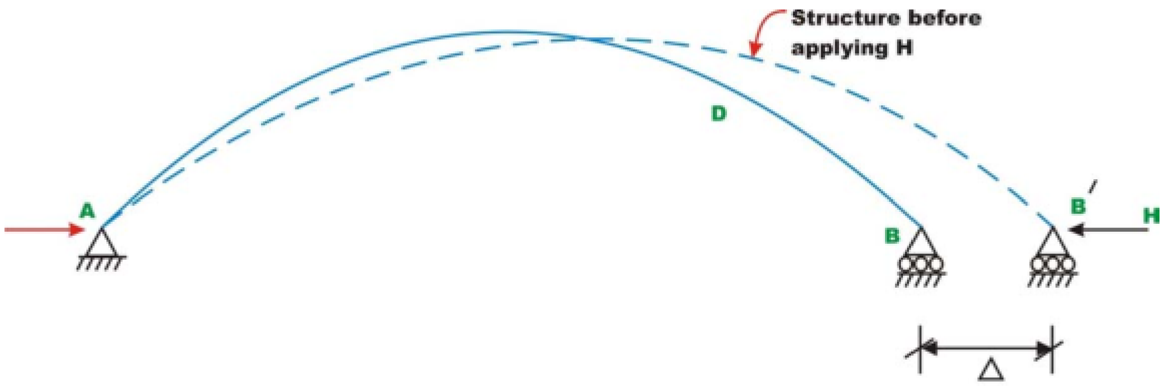

Structure before
applying H
D
A
B'
H
B
Δ

Fig. 33.2c.

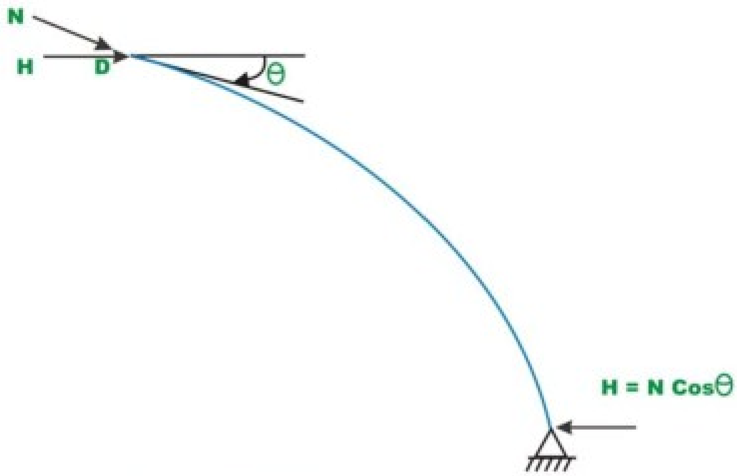


Fig. 33.2d.

From Fig. 33.2b and Fig 33.2c, the bending moment at any cross section of the arch (say D), may be written as

$$M = M_0 - H(h - y) \tag{33.5}$$

The axial compressive force at any cross section (say D) may be written as

$$N = N_0 + H\cos\theta \tag{33.6}$$

Where θ is the angle made by the tangent at D with horizontal (vide Fig 33.2d). Substituting the value of M and N in the equation (33.4),

$$\frac{\partial U}{\partial H} = 0 = -\int_0^s \frac{M_0 - H(h - y)}{EI}(h - y)ds + \int_0^s \frac{N_0 + H\cos\theta}{EA}\cos\theta\, ds \tag{33.7a}$$

Let, $\tilde{y} = h - y$

$$-\int_0^s \frac{M_0 - H\tilde{y}}{EI}\tilde{y}\, ds + \int_0^s \frac{N_0 + H\cos\theta}{EA}\cos\theta\, ds = 0 \tag{33.7b}$$

Solving for H , yields

$$-\int_0^s \frac{M_0}{EI}\tilde{y}\,ds + \int_0^s \frac{H\tilde{y}^2}{EI}ds + \int_0^s \frac{N_0}{EA}\cos\theta\,ds + \int_0^s \frac{H\cos^2\theta}{EA}ds = 0$$

$$H = \frac{\int_0^s \frac{M_0}{EI}\tilde{y}\,ds - \int_0^s \frac{N_0}{EA}\cos\theta\,ds}{\int_0^s \frac{\tilde{y}^2}{EI}ds + \int_0^s \frac{\cos^2\theta}{EA}ds} \tag{33.8}$$

Using the above equation, the horizontal reaction H for any two-hinged symmetrical arch may be calculated. The above equation is valid for any general type of loading. Usually the above equation is further simplified. The second term in the numerator is small compared with the first terms and is neglected in the analysis. Only in case of very accurate analysis second term s considered. Also for flat arched, $\cos\theta \cong 1$ as θ is small. The equation (33.8) is now written as,

$$H = \frac{\int_0^s \frac{M_0}{EI}\tilde{y}\,ds}{\int_0^s \frac{\tilde{y}^2}{EI}ds + \int_0^s \frac{ds}{EA}} \tag{33.9}$$

As axial rigidity is very high, the second term in the denominator may also be neglected. Finally the horizontal reaction is calculated by the equation

$$H = \frac{\int_0^s \frac{M_0}{EI}\tilde{y}\,ds}{\int_0^s \frac{\tilde{y}^2}{EI}ds} \tag{33.10}$$

For an arch with uniform cross section EI is constant and hence,

$$H = \frac{\int_0^s M_0\tilde{y}\,ds}{\int_0^s \tilde{y}^2\,ds} \tag{33.11}$$

In the above equation, M_0 is the bending moment at any cross section of the arch when one of the hinges is replaced by a roller support. $\tilde{y}$ is the height of the arch as shown in the figure. If the moment of inertia of the arch rib is not constant, then equation (33.10) must be used to calculate the horizontal reaction H .

33.2.2 Temperature effect

Consider an unloaded two-hinged arch of span L. When the arch undergoes a uniform temperature change of $T°C$, then its span would increase by $\alpha L T$ if it were allowed to expand freely (vide Fig 33.3a). α is the co-efficient of thermal expansion of the arch material. Since the arch is restrained from the horizontal movement, a horizontal force is induced at the support as the temperature is increased.

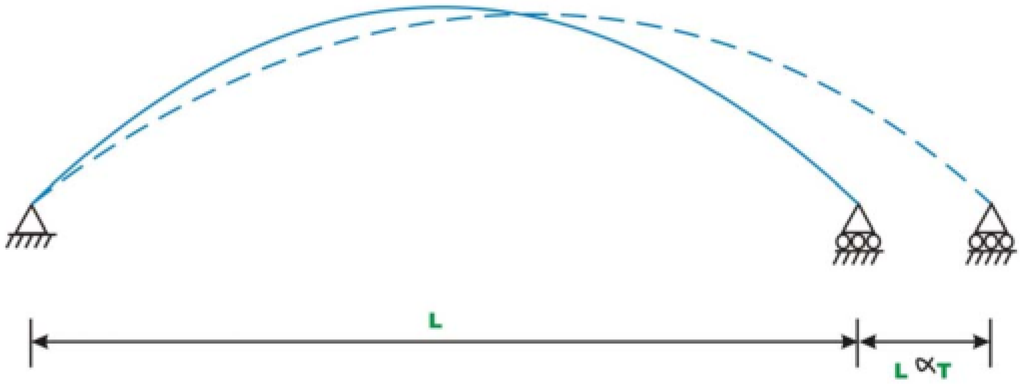
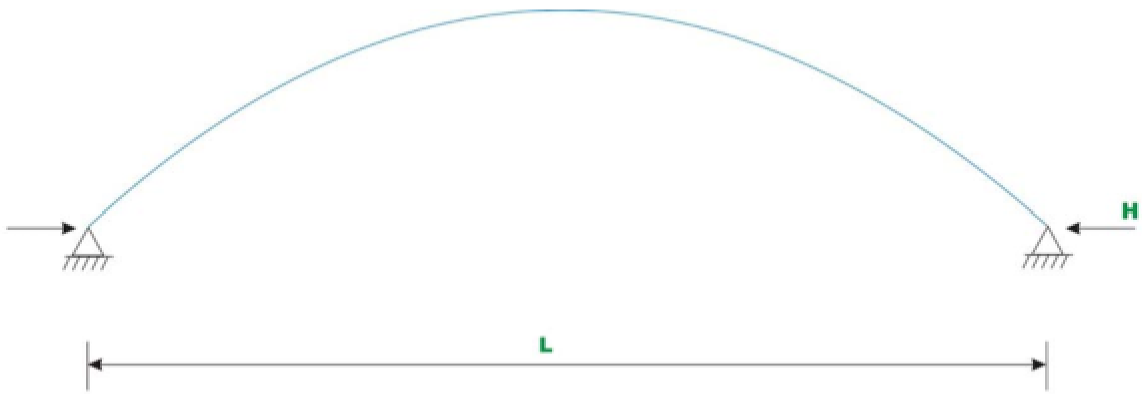


Fig. 33.3a



Now applying the Castigliano's first theorem,

$$\frac{\partial U}{\partial H} = \alpha L T = \int_0^s \frac{H\tilde{y}^2}{EI} ds + \int_0^s \frac{H\cos^2\theta}{EA} ds$$

Solving for H,

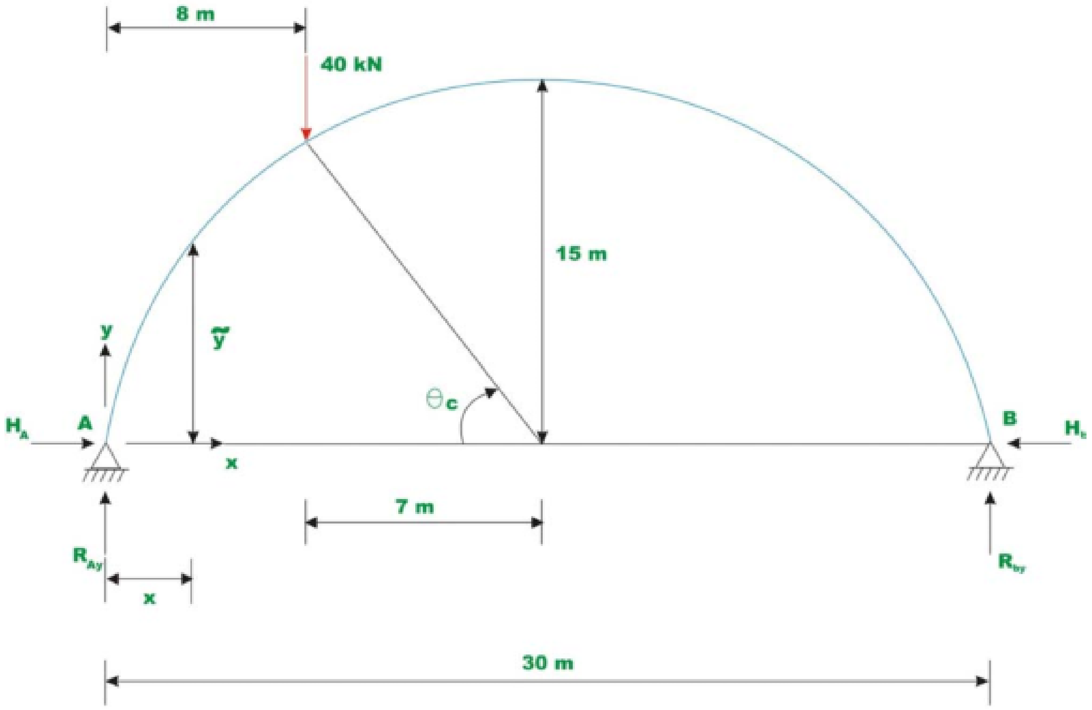
$$H = \frac{\alpha LT}{\int_0^s \frac{\tilde{y}^2}{EI} ds + \int_0^s \frac{\cos^2\theta}{EA} ds} \quad (33.13)$$

The second term in the denominator may be neglected, as the axial rigidity is quite high. Neglecting the axial rigidity, the above equation can be written as

$$H = \frac{\alpha LT}{\int_0^s \frac{\tilde{y}^2}{EI} ds} \quad (33.14)$$

Example 33.1

A semicircular two hinged arch of constant cross section is subjected to a concentrated load as shown in Fig 33.4a. Calculate reactions of the arch and draw bending moment diagram.

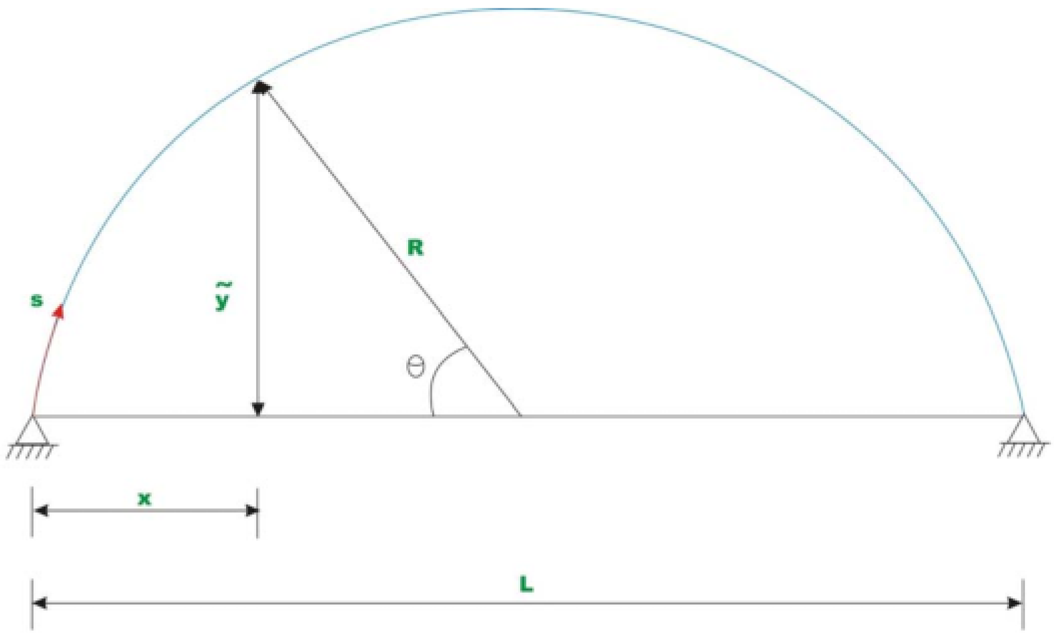


Solution:

Taking moment of all forces about hinge B leads to,

$$R_{ay} = \frac{40 \times 22}{30} = 29.3\ \ 3\ \text{ kN } (\uparrow)$$

$$\sum Fy = 0 \qquad \Rightarrow R_{by} = 10.67 \text{ kN } (\uparrow) \qquad (1)$$



From Fig. 33.4b,

$$\tilde{y} = R \sin\theta$$

$$x = R(1 - \cos\theta)$$

$$ds = R\, d\theta \qquad (2)$$

$$\tan\theta_c = \frac{13.267}{7} \qquad \Rightarrow \theta_c = 62.18° = \pi/_{2.895}\, rad$$

Now, the horizontal reaction H may be calculated by the following expression,

$$H = \frac{\int_0^s M_0 \tilde{y}\, ds}{\int_0^s \tilde{y}^2\, ds} \qquad (3)$$

Now M_0 the bending moment at any cross section of the arch when one of the hinges is replaced by a roller support is given by,

$$M_0 = R_{ay}x = R_{ay}\, R(1-\cos\theta) \qquad\qquad 0 \le \theta \le \theta_c$$

and,

$$\begin{aligned} M_0 &= R_{ay}\, R(1-\cos\theta) - 40(x-8) \\ &= R_{ay}\, R(1-\cos\theta) - 40\{R(1-\cos\theta)-8\} \qquad \theta_c \le \theta \le \pi \end{aligned} \qquad (4)$$

Integrating the numerator in equation (3),

$$\int_0^s M_0 \tilde{y}\, ds = \int_0^{\theta_c} R_{ay} R^3 (1-\cos\theta)\sin\theta\, d\theta + \int_{\theta_c}^{\pi} [R_{ay}\, R(1-\cos\theta) - 40\{R(1-\cos\theta)-8\}] R\sin\theta\, R d\theta$$

$$= R_{ay} R^3 \int_0^{\pi/2.895} (1-\cos\theta)\sin\theta\, d\theta + R^2 \int_{\pi/2.895}^{\pi} [R_{ay}\, R(1-\cos\theta)\sin\theta - 40\{R(1-\cos\theta)\sin\theta - 8\sin\theta\}] d\theta$$

$$= R_{ay} R^3 [-\cos\theta]_0^{\pi/2.895} + R^2 \Big[[R_{ay}\, R(-\cos\theta)]_{\pi/2.895}^{\pi} - [40\, R(-\cos\theta)]_{\pi/2.895}^{\pi} + [40 \times 8(-\cos\theta)]_{\pi/2.895}^{\pi} \Big]$$

$$= 0.533 R_{ay} R^3 + R^2 [[1.4667 R_{ay}\, R] - [40\, R(1.4667)] + [40 \times 8(1.4667)]]$$

$$= 52761.00 + 225(645.275 - 410.676) = 105545.775 \qquad (5)$$

The value of denominator in equation (3), after integration is,

$$\begin{aligned} \int_0^s \tilde{y}^2 ds &= \int_0^{\pi} (R\sin\theta)^2 R d\theta \\ &= R^3 \int_0^{\pi} \left(\frac{1-\cos 2\theta}{2} \right) d\theta = R^3 \left(\frac{\pi}{2} \right) = 5301.46 \end{aligned} \qquad (6)$$

Hence, the horizontal thrust at the support is,

$$H = \frac{105545.775}{5301.46} = 19.90 \quad \text{kN} \qquad (7)$$

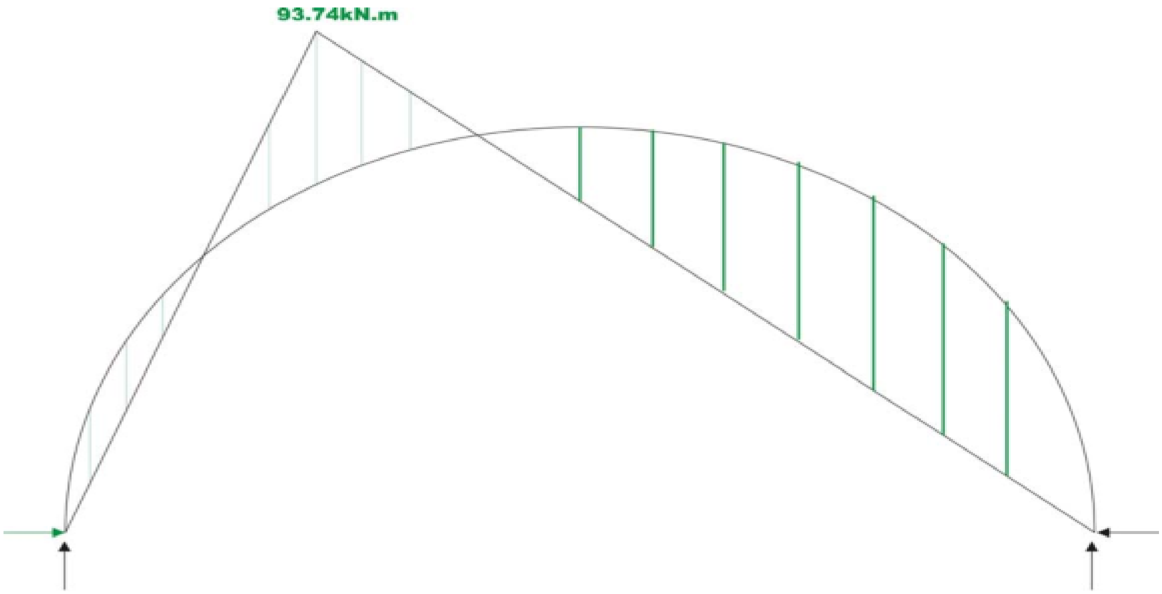
Bending moment diagram

Bending moment M at any cross section of the arch is given by,

$$\begin{aligned} M &= M_0 - H\tilde{y} \\ &= R_{ay} R(1-\cos\theta) - HR\sin\theta \qquad\qquad 0 \le \theta \le \theta_c \qquad (8) \\ &= 439.95(1-\cos\theta) - 298.5\sin\theta \end{aligned}$$

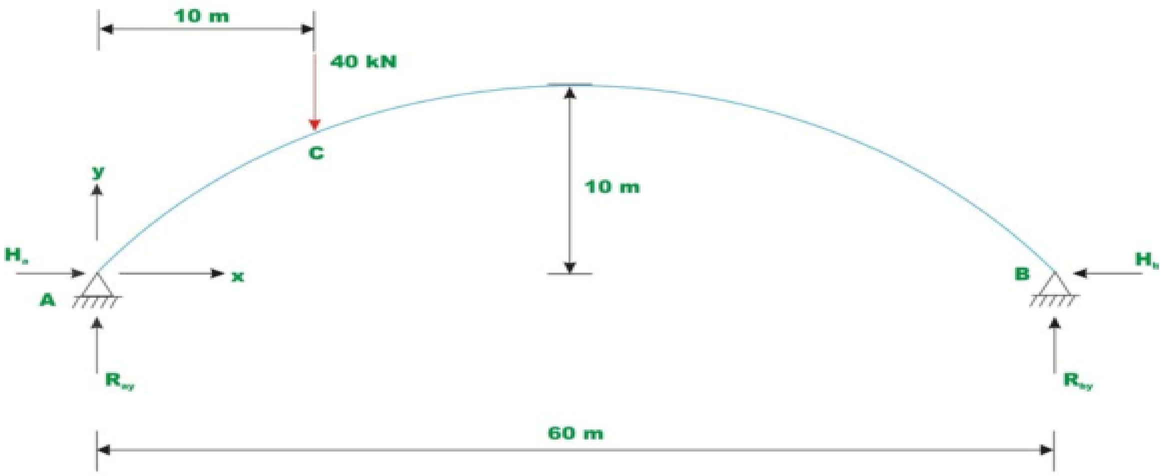
$$M = 439.95(1-\cos\theta) - 298.5\sin\theta - 40(15(1-\cos\theta) - 8) \qquad \theta_c \le \theta \le \pi \qquad (9)$$

Using equations (8) and (9), bending moment at any angle θ can be computed. The bending moment diagram is shown in Fig. 33.4c.



Example 33.2

A two hinged parabolic arch of constant cross section has a span of 60m and a rise of 10m. It is subjected to loading as shown in Fig.33.5a. Calculate reactions of the arch if the temperature of the arch is raised by $40°C$. Assume co-efficient of thermal expansion as $\alpha = 12 \times 10^{-6} / °C$.



Taking A as the origin, the equation of two hinged parabolic arch may be written as,

$$y = \frac{2}{3}x - \frac{10}{30^2}x^2 \qquad (1)$$

The given problem is solved in two steps. In the first step calculate the horizontal reaction due to $40\ \text{kN}$ load applied at C. In the next step calculate the horizontal reaction due to rise in temperature. Adding both, one gets the horizontal reaction at the hinges due to combined external loading and temperature change. The horizontal reaction due to $40\ \text{kN}$ load may be calculated by the following equation,

$$H_1 = \frac{\int_0^s M_0 y\, ds}{\int_0^s \tilde{y}^2\, ds} \qquad (2a)$$

For temperature loading, horizontal reaction is given by,

$$H_2 = \frac{\alpha L T}{\int_0^s \frac{y^2}{EI} ds} \qquad (2b)$$

Where L is the span of the arch.

For $40\ \text{kN}$ load,

$$\int_0^s M_0 y\, ds = \int_0^{10} R_{ay}\, x y\, dx + \int_{10}^{60} \left[R_{ay} x - 40(x-10)\right] y\, dx \qquad (3)$$

Please note that in the above equation, the integrations are carried out along the *x*-axis instead of the curved arch axis. The error introduced by this change in the variables in the case of flat arches is negligible. Using equation (1), the above equation (3) can be easily evaluated.

The vertical reaction A is calculated by taking moment of all forces about B. Hence,

$$R_{ay} = \frac{1}{60}\left[40 \times 50\right] = 33.33 \quad \text{kN}$$

$$R_{by} = 6.67 \quad \text{kN}.$$

Now consider the equation (3),

$$\int_0^l M_0 y\, dx = \int_0^{10} (33.33)x(\frac{2}{3}x - \frac{10}{30^2}x^2)\,dx + \int_{10}^{60} [(33.33)x - 40(x-10)](\frac{2}{3}x - \frac{10}{30^2}x^2)\,dx$$

$$= 6480.76 + 69404.99 = 74885.75 \qquad (4)$$

$$\int_0^l y^2\, dx = \int_0^{60} \left[\frac{2}{3}x - \frac{10}{30^2}x^2\right]^2 dx$$

$$= 3200 \qquad (5)$$

Hence, the horizontal reaction due to applied mechanical loads alone is given by,

$$H_1 = \frac{\int_0^l M_0 y \quad dx}{\int_0^l y^2 \quad dx} = \frac{75885.75}{3200} = 23.71 \quad \text{kN} \qquad (6)$$

The horizontal reaction due to rise in temperature is calculated by equation (2b),

$$H_2 = \frac{12 \times 10^{-6} \times 60 \times 40}{3200 / EI} = \frac{EI \times 12 \times 10^{-6} \times 60 \times 40}{3200}$$

Taking $E = 200 \quad \text{kN/mm}^2$ and $I = 0.0333\, m^4$

$$H_2 = 59.94 \quad \text{kN}. \qquad (7)$$

Hence the total horizontal thrust $H = H_1 + H_2 = 83.65$ kN.

When the arch shape is more complicated, the integrations $\int_0^s \frac{M_0 y}{EI} ds$ and $\int_0^s \frac{y^2}{EI} ds$ are accomplished numerically. For this purpose, divide the arch span in to n equals divisions. Length of each division is represented by $(\Delta s)_i$ (vide Fig.33.5b). At the midpoint of each division calculate the ordinate y_i by using the equation $y = \frac{2}{3}x - \frac{10}{30^2}x^2$. The above integrals are approximated as,

$$\int_0^s \frac{M_0 y}{EI} ds = \frac{1}{EI} \sum_{i=1}^{n} (M_0)_i y_i (\Delta s)_i \qquad (8)$$

$$\int_0^s \frac{y^2}{EI} ds = \frac{1}{EI} \sum_{i=1}^{n} (y)_i^2 (\Delta s)_i \qquad (9)$$

The complete computation for the above problem for the case of external loading is shown in the following table.

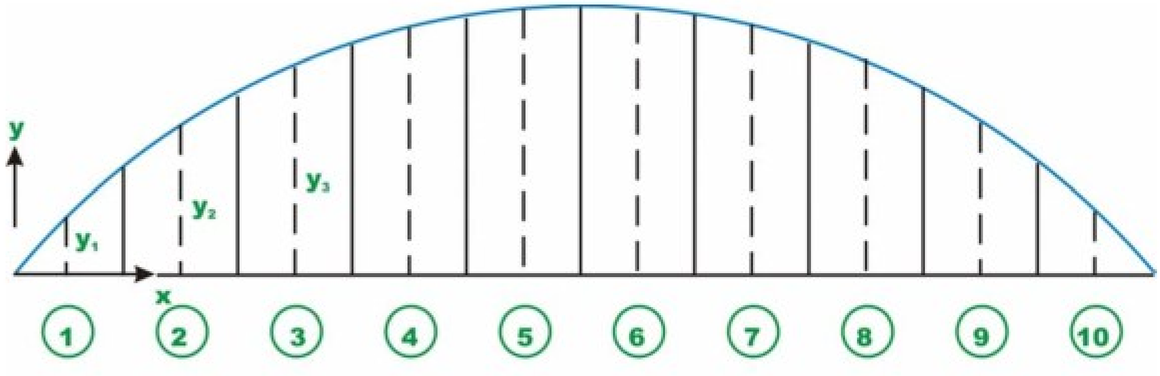


Table 1. Numerical integration of equations (8) and (9)

Segment No	Horizontal distance *x* Measured from A (m)	Corresponding y_i (m)	Moment at that Point $(M_0)_i$ (kNm)	$(M_0)_i y_i (\Delta s)_i$	$(y)_i^2 (\Delta s)_i$
1	3	1.9	99.99	1139.886	21.66
2	9	5.1	299.97	9179.082	156.06
3	15	7.5	299.95	13497.75	337.5
4	21	9.1	259.93	14192.18	496.86
5	27	9.9	219.91	13062.65	588.06
6	33	9.9	179.89	10685.47	588.06
7	39	9.1	139.87	7636.902	496.86
8	45	7.5	99.85	4493.25	337.5
9	51	5.1	59.83	1830.798	156.06
10	57	1.9	19.81	225.834	21.66
			$\sum$	75943.8	3300.3

$$H_1 = \frac{\sum (M_0)_i y_i (\Delta s)}{\sum (y)_i^2 (\Delta s)_i} = \frac{75943.8}{3200.3} = 23.73 \text{ kN} \qquad (10)$$

This compares well with the horizontal reaction computed from the exact integration.